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ON THE STRONG LAW OF LARGE NUMBERS FOR THE FAMILY OF FIRST PASSAGE TIMES OF THE PARABOLIC BOUNDARY BY THE MARKOV RANDOM WALK, DESCRIBED BY THE AUTOREGRESSIVE PROCESS AR(1)

F.G. Rahimov[1,2], G.A. Baghirova[1], V.S. Khalilov[1], U.F. Mammadova[1]

[1]Baku State University, Department of Probability Theory and Mathematical Statistics, Baku, Azerbaijan
[2]Baku State University, Institute of Applied Mathematics, Baku, Azerbaijan
e-mail: ragimovf@rambler.ru, bagirova.gulnara.amir@bsu.edu.az

Abstract. In the paper the strong law of large numbers for the family of first passage times of the parabolic boundary by the Markov random walk, described by the autoregressive process AR(1) is proved.

Keywords: Strong law of large numbers, first-order autoregressive process, Markov random walk, first passage time.

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1. Introduction

Mathematical models based on stochastic processes and probabilistic methods have found numerous applications in engineering, economics and industrial optimization. In particular, mathematical models for optimal resource allocation and optimal distribution problems have been investigated in [1,3], while probabilistic methods related to distributional properties have been studied in [2]. These applications demonstrate the importance of developing new limit theorems for stochastic models such as the Markov random walk considered in the present paper.

We consider the first-order autoregressive process AR(1) in the form

$$X_n - m = \beta(X_{n-1} - m) + \xi_n, \qquad n \geq 1\,, \tag{1}$$

where $m, \beta \in (-\infty, \infty)$ are fixed constants and ξ_n, $n \geq 1$ is a sequence of independent and identically distributed random variables (innovation) defined on the probability space $(\Omega, \mathcal{F}, P)$.

Note that the process AR(1) of the form (1) is considered in the works [6,10,16] where are studied tests of statistical hypothesis for unknown parameters m and β.

Let

$$L_n = \sum_{k=0}^{n} X_k\,, \qquad n \geq 1$$

$L_n,\ n \geq 1$ is Markov random walk, described by sums of values of the process AR(1) in (1). The definition of a Markov random walk in general case is given in the work [7-9,13,19].

The Markov random walk of the form $L_n = \sum_{k=0}^{n} X_k\,, n \geq 1$ is generalization of the ordinary random walk, described by sums of independent and identically distributed random walk. In the case $\beta = 0$ we have for $n \geq 1$

$$L_n = \sum_{k=1}^{n} \eta_k = S_n\,, \qquad \eta_k = m + \xi_k,$$

where $S_n, n \geq 1$ – ordinary random walk.

Boundary problems for ordinary random walks where studied in works [5,17].

We consider the family of first passage time

$$\tau_a = \inf\{n \geq 1 : L_n > a\sqrt{n}\}\ ,\ a > 0 \qquad (2)$$

In the case $\beta = 0$ $(L_n = S_n)$ there exist many results concerning the family of first passage times (2). The limit theorems (central limit theorem, law of large numbers, and strong law of large numbers) for the family of stopping times (2) in the case $\beta = 0$ were studied in [3,14,17,18].

In present paper we prove the strong law of large numbers for the family $\{\tau_a\ , a \geq 0\}$ defined in (2) for $|\beta| < 1$.

Similar, problems where studied in works [11,12] for linear boundary.

2. Formulation and proof of the main result

We assume that $E\xi_1 = 0,\ 0 < E\xi_1^2 = \sigma^2 < \infty\ \ E|X_0 - m|^2 < \infty,\ |\beta| < 1$ and initial value of process AR(1) X_0 does not depend on the innovation $\{\xi_n\}$.

Under these conditions is well known process AR(1) is stationary and $E|X_n - m|^2 \to \frac{\sigma^2}{1-\beta^2}\ \ as\ n \to \infty$ [10,14,15].

The following theorem is called the strong law of large numbers for the family of first passage times τ_a defined in (2).

Theorem. Let $|\beta| < 1, E\xi_1 = 0, D\xi_1 = \sigma^2 < \infty, E|X_0 - m|^2 < \infty$ and

$$\frac{L_n}{n} \xrightarrow{a.s} m > 0 \qquad as\ n \to \infty \qquad (3)$$

Then

$$\frac{\tau_a}{N_a} \xrightarrow{a.s} \frac{1}{m} \qquad as\ a \to \infty$$

where $N_a = \left(\frac{a}{m}\right)^2$

The proof follows from condition (3)

$$P(\sup_n \frac{L_n}{\sqrt{n}} = \infty) = 1 \tag{4}$$

Hence, by the definition of the first passage time τ_a in (2) we obtain

$$P(\tau_a < \infty) = P\left(\sup_n \frac{L_n}{\sqrt{n}} > a\right) = 1 \quad \text{for all } a \geq 0$$

By the definition of τ_a we have for $n \geq 0$

$$P(\tau_a > n) = P\left(\max_{0 \leq k \leq n} \frac{L_n}{\sqrt{n}} \leq a\right) \tag{5}$$

Letting $a \to \infty$ in (5) we obtain

$$\tau_a \xrightarrow{P} \infty \quad as\ a \to \infty \tag{6}$$

Taking into account that the first passage time τ_a in (2) is an increasing function of , it follows from (6) that

$$\tau_a \xrightarrow{a.s} \infty \quad as\ a \to \infty \tag{7}$$

To prove that

$$\frac{L_{\tau_a}}{\tau_a} \xrightarrow{a.s} m \quad as\ \ a \to \infty \tag{8}$$

Let

$$A = \left\{\omega : \frac{L_n}{n} \to m,\ \ n \to \infty\ \right\},$$
$$B = \{\omega : \tau_a \to \infty,\ a \to \infty\ \},$$

and

$$C = \left\{\omega : \frac{L_{\tau_a}}{\tau_a} \to m,\ \ a \to \infty\ \right\}$$

By condition (3) $P(A) = 1$ and by (7) $P(B) = 1$.Clearly $A \cap B \subset C$ and $P(A \cap B) = 1$ hence, $P(C) = 1$.

Thus, (8) holds.

Next, by the definition of the first passage time τ_a in (2) we have

$$\frac{L_{\tau_a - 1}}{\tau_a} \leq \frac{a\sqrt{\tau_a}}{\tau_a} < \frac{L_{\tau_a}}{\tau_a}$$

or

$$\frac{\tau_a}{L_{\tau_a - 1}} \leq \frac{\sqrt{\tau_a}}{a} < \frac{\tau_a}{L_{\tau_a}} \tag{9}$$

Using (7) , (8) and (9) we complete the proof of the theorem.

Remark. Note that under these conditions $|\beta| < 1\ \ E|X_0 - m|^2 < \infty$, $E\xi_1 = 0, D\xi_1 = \sigma^2 < \infty$ hold

$$\frac{1}{n}\sum_{k=0}^{n} X_k \xrightarrow{P} m \tag{10}$$

as can be to seen from (1)

$$\sum_{k=1}^{n}(X_k - m) = \beta\sum_{k=1}^{n}(X_{k-1} - m) + \sum_{k=1}^{n}\xi_k$$

or

$$\frac{(1-\beta)}{n}\sum_{k=1}^{n}(X_k-m)=\frac{X_0-X_n}{n}+\frac{1}{n}\sum_{k=1}^{n}\xi_k \qquad (11)$$

By the strong law of large numbers for the random variables ξ_n we have

$$\frac{1}{n}\sum_{k=1}^{n}\xi_k \xrightarrow{a.s} 0 \qquad as\ n\to\infty \qquad (12)$$

clearly

$$\frac{X_0}{n}\xrightarrow{a.s} 0 \qquad as\ n\to\infty \qquad (13)$$

to prove

$$\frac{X_n}{n}\xrightarrow{P} 0 \qquad as\ n\to\infty$$

from (1), by iteration, we obtain

$$X_n-m=\beta^n(X_0-m)+\sum_{k=0}^{n-1}\beta^k\,\xi_{n-k} \qquad (14)$$

Taking into account that variables X_0 and ξ_n are independent, we obtain from (11)

$$E|X_n-m|^2\to\sigma^2\sum_{k=0}^{\infty}\beta^{2k}=\frac{\sigma^2}{1-\beta^2} \qquad (15)$$

Then, by Markov`s inequality

$$P(|X_n|>n\varepsilon)\le\frac{EX_n^2}{n^2\varepsilon^2} \qquad (16)$$

It follows from (12) that $EX_n^2=O(1)$.

Thus, from (13), we have

$$.\frac{X_n}{n}\xrightarrow{P} 0 \qquad as\ n\to\infty \qquad (17)$$

İt follows from (17) , (13) , (12) and (11) that (10) holds.

Note that , if $\frac{X_n}{n}\xrightarrow{a.s} 0 \quad as\ n\to\infty$ from (11) follows (3).

3. Conclusion

In this paper, we consider the family of first passage times for a parabolic boundary by a Markov random walk, defined by the sums of a first-order autoregressive (AR(1)) process. The strong law of large numbers is proved for this family of the first passage times.

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